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Ellis L. Johnson

OPERATIONS RESEARCH CENTER

INSTITUTE OF ENGINEERING RESEARCH

UNIVERSITY OF CALIFORNIA - BERKELEY

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A PROOF OF FOUR-COLORING THE EDGES OF A REGULAR THREE-DEGREE GRAPH

A graph G consists of a finite set V of points called vertices together with a set E of unordered pairs of vertices called edges. Note that this definition does not permit multiple edges or loops. A subgraph G' of a graph G has vertices $V' \subseteq V$ and edges $E' \subseteq E$, where $[v_1, v_2] \in E'$ only for $v_1 \in V'$ and $v_2 \in V'$. Coloring the edges of a graph means assigning to each edge a color so that no vertex is incident to two edges of the same color. A simple path is a finite sequence of distinct vertices $v_1, \ldots, v_n$ such that $[v_i, v_{i+1}] \in E$ for $i = 1, \ldots, n-1$. A connected graph has a simple path between every pair of vertices, and a graph not connected consists of connected components. In a connected graph, a bridge is an edge such that its removal from E causes G to not be connected. The degree of a vertex is the number of edges incident to that vertex, and a regular graph has the same degree at each vertex.

Coloring the edges of a regular 3-degree graph is of special interest because the 4-color conjecture is equivalent [4] to 3-coloring the edges of a planar, regular 3-degree graph which is connected and without bridges. Edges of a connected, regular 3-degree graph can easily be colored with 5 colors and can be colored with 4 colors provided there is no bridge [1]. Theorem 2 states that the edges of any regular, 3-degree graph can be colored with 4 colors. This result also follows directly from a result on coloring the nodes of a graph [2], but the method used here is of interest because it gives an algorithm whose length depends linearly on the size of the graph, and it is used to characterize the regular 3-degree graphs in an inductive manner as given in Theorem 1. Theorem 2 does not depend on Theorem 1. The method is similar to "splits" first used by Frink [3].

A regular 3-degree graph having n vertices must have $\frac{3}{2}n$ edges, and hence n must be even. There is one such graph having 4 vertices and two having 6 vertices. They are shown below with a 3-coloring of the edges:

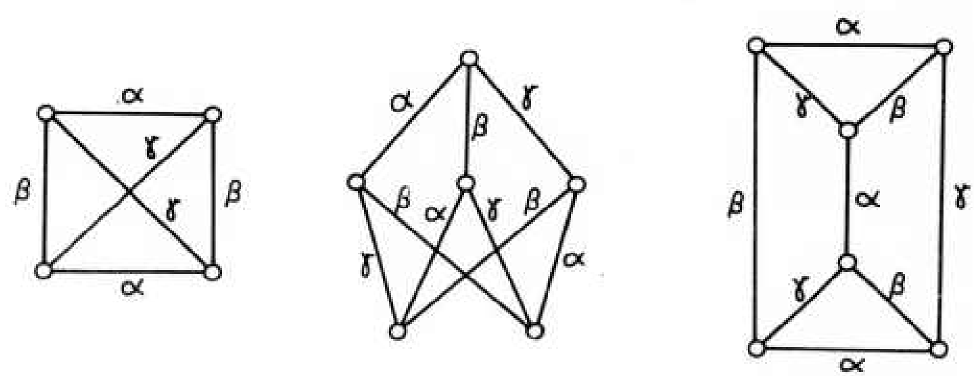

FIGURE 1

An H-tree is defined as a graph with 5 edges and 6 vertices having degrees 3, 3, 1, 1, 1, and 1.

<u>LEMMA 1</u>. Any connected, regular graph of degree 3 with n vertices, $n \geq 6$, has an H-tree as a subgraph.

<u>PROOF</u>: For any vertex v_0, there are three distinct edges incident to v_0, say $e_1 = [v_1, v_0]$, $e_2 = [v_0, v_2]$, $e_3 = [v_0, v_3]$. Consider the edges with both ends in $\{v_1, v_2, v_3\}$.

There cannot be 3 such edges, because there can be no connected component with 4 vertices. Consider the case of 2 such edges, say $e_4 = [v_1, v_2]$ and $e_5 = [v_2, v_3]$ (Figure 2(a)). Then v_1 is incident to another edge $e_6 = [v_1, v_4]$, v_4 different from v_0, v_1, v_2, and v_3. There are two more edges incident to v_4, $e_7 = [v_4, v_5]$ and $e_8 = [v_4, v_6]$. Furthermore v_5 and v_6 are different from v_0, v_1, v_2 because they already have degree 3. Hence e_1, e_4, e_6, e_7, and e_8 constitute an H-tree.

Consider the case of exactly one edge, say $e_4 = [v_2, v_3]$, with both ends in $\{v_1, v_2, v_3\}$ (Figure 2(b)). Then v_1 is incident to two other edges, $e_5 = [v_1, v_4]$ and $e_6 = [v_1, v_5]$. Furthermore, v_4 and v_5 are different from v_0, v_1, v_2, and v_3 because there is no edge from v_1 to v_2 or v_3. Hence e_1, e_2, e_3, e_5, e_6 form an H-tree. In case there is no edge with both ends in $\{v_1, v_2, v_3\}$, v_1 is incident to two other edges, $e_4 = [v_1, v_4]$ and $e_5 = [v_1, v_5]$, and then e_1, e_2, e_3, e_4, e_5 form an H-tree (Figure 2(c)). Thus lemma 1 is proven.

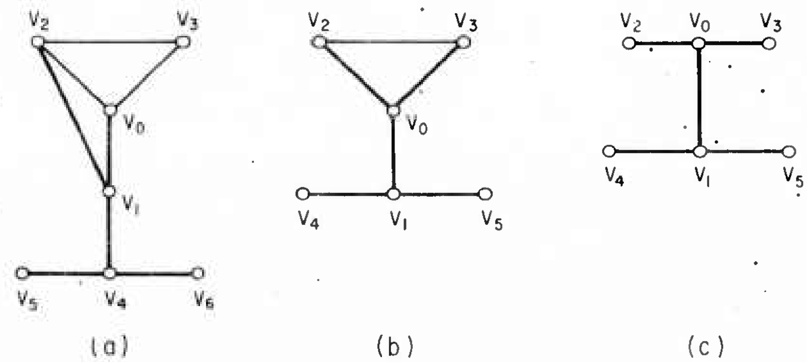

FIGURE 2

Consider an H-tree in G with vertices v_0 , v_1 , v_2 , v_3 , v_4 , v_5 and edges $e_1 = [v_0 , v_1]$, $e_2 = [v_0 , v_2]$, $e_3 = [v_0 , v_3]$, $e_4 = [v_1 , v_4]$, $e_5 = [v_1 , v_5]$ (Figure 2(c)). The H-tree is called acceptable if no two edges of G with one end in $\{v_2 , v_3\}$ and the other end in $\{v_4 , v_5\}$ are incident to the same vertex.

<u>LEMMA 2</u>. Any connected regular graph G of degree 3 on n vertices, $n \geq 8$, has an acceptable H-tree as a subgraph.

<u>PROOF</u>: From lemma 1 there is an H-tree in G . Denote the vertices and edges as in Figure 2(c) and consider the edges with one end in $\{v_2 , v_3\}$ and the other in $\{v_4 , v_5\}$. There cannot be 4 such edges because then there would be a connected component on 6 vertices and $n \geq 8$ for G .

Consider first the case of 3 such edges, say $[v_2 , v_4]$ is not an edge of G (Figure 3(a)). Let $e_6 = [v_2 , v_5]$, and let $e_7 = [v_2 , v_6]$ be the other edge incident to v_2 . Let $e_8 = [v_6 , v_7]$ and $e_9 = [v_6 , v_8]$ be the other two edges incident to v_6 . Then v_7 and v_8 are different from v_0 , v_1 , v_2 , v_3 , v_5 , because they already have degree 3. Then e_2 , e_6 , e_7 , e_8 , e_9 form an acceptable H-tree, because all the edges with one end in $\{v_0 , v_5\}$ have the other end in $\{v_1 , v_2 , v_3\}$.

Consider now the case (Figure 3(b)) of 2 such edges and suppose they are incident to the same vertex, say $[v_3 , v_4]$, $[v_3 , v_5]$. If they are not incident to the same vertex, or if there is only one such edge, then the H-tree is already acceptable. Now v_2 is incident to two other edges $e_6 = [v_2 , v_6]$, $e_7 = [v_2 , v_7]$, where v_6 and v_7 are different from v_0 , v_1 , v_2 , v_3 , v_4 , and v_5 . Then e_1 , e_2 , e_3 , e_6 , e_7 form an acceptable H-tree because all the edges with one end in $\{v_1 , v_3\}$ have the other in $\{v_0 , v_4 , v_5\}$.

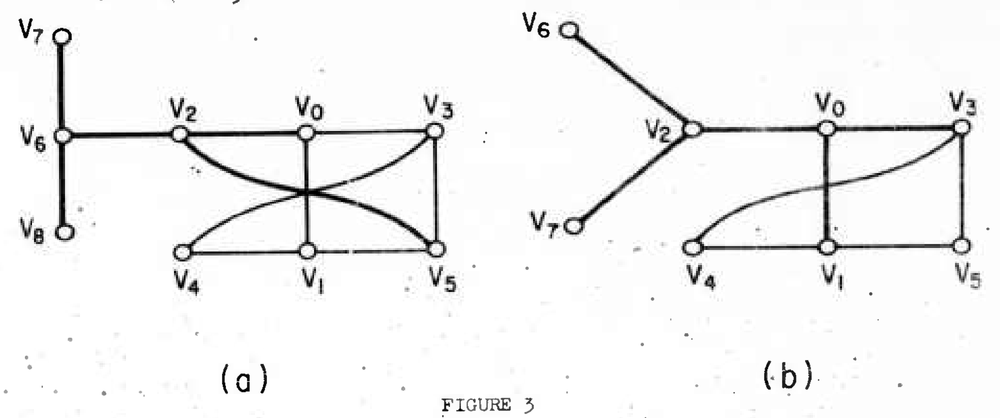

FIGURE 3

Let an acceptable H-tree in G be denoted as in the definition and Figure 2(c). A new graph G' , called the H-reduced graph, can be formed by deleting v_0 and v_1 from V and e_1 , e_2 , e_3 , e_4 , e_5 from E and adjoining either $e_1' = [v_2 , v_4]$ and $e_2' = [v_3 , v_5]$ or $e_1'' = [v_2 , v_5]$ and $e_2'' = [v_3 , v_4]$ to E depending on which pair is not in E . In the proofs of theorems 1 and 2, an acceptable H-tree in G will be assumed to be denoted so that e_1' and e_2' are adjoined to form G' .

For any graph G having two edges $e_1 = [v_2 , v_4]$, $e_2 = [v_3 , v_5]$ with v_2 , v_3 , v_4 , v_5 distinct vertices, a new graph G' , called the H-enlarged graph can be found by deleting e_1 and e_2 from E , and adjoining v_0' and v_1' to V and either the five edges $[v_0' , v_1']$, $[v_0' , v_2]$, $[v_0' , v_3]$, $[v_1' , v_4]$ and $[v_1' , v_5]$ or the five edges $[v_0' , v_1']$, $[v_0' , v_2]$, $[v_0' , v_5]$, $[v_1' , v_3]$, $[v_1' , v_4]$. Either enlargement can always be done provided v_2 , v_3 ,

v_4 , v_5 are distinct vertices.

THEOREM 1. For $n \geq 6$, every connected regular 3-degree graph G on $n + 2$ vertices is an H-enlargement of a connected, regular 3-degree graph G' on n vertices.

PROOF: From lemma 2, there is an acceptable H-tree in G . Clearly the reduced graph G' is also a regular 3-degree graph. Suppose G' is not connected. Then there are two vertices v and v' of G' with no simple path between them. By G connected, there is such a simple path in G . The path must include v_0 or v_1 because otherwise it is a path in G' .

Case 1. Suppose it includes only v_0 (Figure 4(a)). Then there is no simple path from v_2 to v_3 in G that does not include v_0 or v_1 because if there were, we could use it in place of v_2 , v_0 , v_3 and find a simple path from v to v' in G'. Hence, there is no simple path in G from v_2 to v_5 , or from v_3 to v_4 , or from v_4 to v_5 that does not include v_0 or v_1 .

Case 2. Suppose now the path includes v_0 and v_1 , say v_0 appears first (Figure 4(b)). If v_0 and v_1 are not adjacent in the path, we can omit the vertices in between to obtain another simple path from v to v' in G with v_0 and v_1 adjacent. Let v_2 be the vertex before v_0 . Then if the path has v_2 , v_0 , v_1 , v_4 , we can form a path in G' by using v_2 , v_4 in place of v_2 , v_0 , v_1 , v_4 . Hence the path has v_2 , v_0 , v_1 , v_5 . Now, just as in case 1, there is no simple path in G from v_2 to v_5 that does not include v_0 or v_1, and hence there is none from v_2 to v_3, or from v_3 to v_4, or from v_4 to v_5 that does not include v_0 or v_1.

Hence, in either case $[v_2, v_5]$ and $[v_3, v_4]$ are not edges of G. Consider the H-reduction in G using the same H-tree and adjoining the edges $[v_2, v_5]$ and $[v_3, v_4]$ instead of $[v_2, v_4]$ and $[v_3, v_5]$. This H-reduction is the same as the preceding one with v_4 and v_5 interchanged. Hence, if this H-reduced graph is also not connected, then there is no simple path in G from v_2 to v_4 or from v_3 to v_5 not including v_0 or v_1.

Therefore, if neither H-reduced graph is connected, then there is no simple path in G from v_2 to v_3, v_4, or v_5 not including v_0 and v_1.

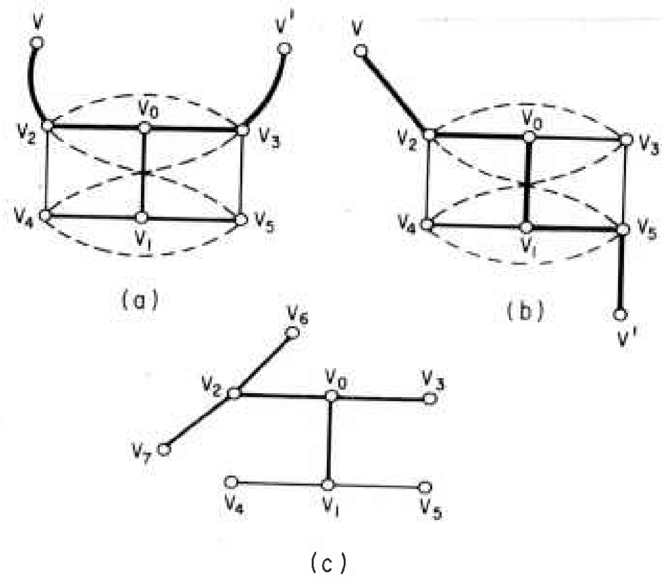

FIGURE 4

Consider, then, the other two edges at v_2 in G, say $e_6 = [v_2, v_6]$ and $e_7 = [v_2, v_7]$. By the above, v_6 and v_7 are different from v_0, v_1, v_2, v_3, v_4, and v_5, and there are no edges from v_6 or v_7 to v_0, v_1, v_3, v_4, or v_5. Hence, e_1, e_2, e_3, e_6, e_7 form an acceptable H-tree (Figure 4(c)). If neither of its reduced graphs is connected, we can continue along the other two edges of v_6. In this way we can continue as long as no connected reduced graph is found, and at each step two new vertices will be found because at every step there is no simple path from the new vertices to any of the old ones except using edges already encountered. But the vertices of G are finite in number, so eventually a connected reduced graph G' is found.

THEOREM 2. The edges of any regular 3-degree graph G can be colored with 4 colors.

PROOF: The theorem is true for G having 4 or 6 vertices (Figure 1). Suppose the theorem is false. Then there is a smallest graph G for which it fails. G is connected because if not, one of its components, which has fewer vertices than G, cannot be 4-colored. G has 8 or more vertices, so by lemma 3 has an acceptable H-tree. Note that G' need not be connected, so only methods of lemmas 1 and 2 need be used to find an acceptable H-tree. The edges of the reduced graph G' can be 4-colored. Denote the colors by α, β, γ, δ.

Suppose $e_1' = [v_2, v_4]$ and $e_2' = [v_3, v_5]$ are different colors, say α and β (Figure 5(a)), in the coloring of G'. Then in G color all the edges except e_1, e_2, e_3, e_4, e_5 the same color as in G'. Color e_2 and e_4 α, and e_3 and e_5 β. Color e_1 γ.

Suppose e_1' and e_2' are colored the same color, say α.

Then color e_2 and e_4 α . Now color e_3 , the color not incident to v_3, and color e_5 the color not incident to v_5 . Then e_1 can be colored the fourth color not used to color e_2 , e_3 , e_4 , e_5 (Figure 5(b)) .

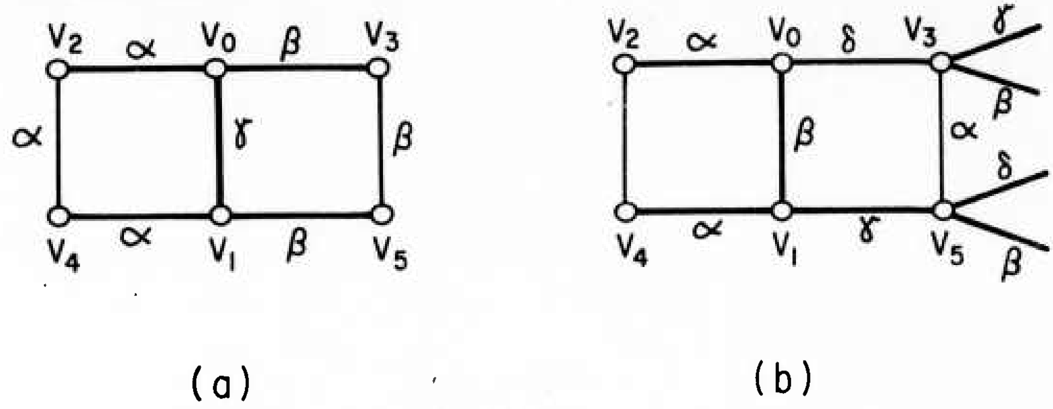

FIGURE 5

REFERENCES

1. Berge, C., Theorie des Graphes et ses Applications, Dunod, Paris (1958).

2. Brooks, R. L., "On Colouring the Nodes of a Network," Cambridge Philos. Soc., v. 37 (1941) pp. 194-197.

3. Frink, O., "A Proof of Peterson's Theorem," Ann. Math., v. 27 (1926) pp. 491-493.

4. Ringel, G., Färbungsprobleme auf Flächen und Graphen , Deutscher Verlag der Wissenschaften (1959).

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